ON REVEALED DIVERSITY

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Abstract. We introduce and characterize axiomatically a diversity criterion, capturing individual dissimilarity as 'revealed' by the different best-choices that members of a society select from a set of opportunities. Diversity ordering is induced by a class of frequency-based evaluation functions, the one element of which is the celebrated diversity measure, Shannon entropy.

JEL classification: D31; D63; I31.

Key words: Diversity, Freedom of Choice, Entropy, Co-cardinality.

"The prospects of peace, tolerance, freedom and democracy in the contemporary world may well lie in the recognition of the plurality of our identities, where personal identity must be understood as an extension of one's own choice of being someone or doing something" (Sen (2006))

1. Introduction

A liberal tradition (see Mill (1859) and Nozick (1974) among others) regards the diversity of a society as a desirable characteristic in itself and considers the freedom of choice of individuals as an adequate tool for guaranteeing such diversity. *How can we measure the diversity of individual choices in a free society*? We answer this question by proposing and characterizing axiomatically a diversity ranking of choice sets that is grounded theoretically and simple to implement. We introduce a new criterion that evaluates the diversity of the best options freely selected by individuals from a suitable set of opportunities. While most of the current economic literature (see e.g. Barberà, Bossert and Pattanaik (2004) or Gravel (2008)) is concerned with evaluation of the diversity of available options from an opportunity set, we measure the extent to which such options allows the revelation of the diversity of individuals. Indeed, the diversity of options in a set does not guarantee the diversity of personal choices. In a democratic society, individuals are always free to select the same option (which is available to all of them), irrespective of the possible diversification of non-valuable opportunities. On the contrary, a non-democratic society could instead force individual diversification irrespective of their actual preferences. Nevertheless, this is the case in which none of the opportunity sets offer any freedom of choice to individuals (see Jones and Sugden (1982) and Pattanaik and Xu (1990)) as individual liberties are violated.

In the present work we consider the freedom of choice as a pre-requisite to meaningfully evaluate social diversity. The aim is therefore to capture the diversity within a free society by focusing on the 'diversity of *actual* choices', which *reveals* the freedom of individuals to pursue their own *personal life-plans*.

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The concept of *revealed diversity* requires jointly considering a set of opportunities and a preference profile. In our setting, *an opportunity set is a collection of positive-valued options* that may be interpreted as possible individual life-plans, rather than different curricula of a schooling system. In other words, an option can be regarded as a bundle of rights and basic liberties (or functionings *à la Sen*)[1], that everybody may claim to have without preventing others from claiming them as well. Thus, opportunities are seen as both *non-rival* and *excludible.* Moreover, all individuals are assumed to be endowed with a well-defined preference ordering when they choose an option from a set of opportunities. As pointed out by Sen (1991), a sensible analysis of diversity cannot disregard the preferences of the individuals concerned for such diversities. Consider for example an individual having a preference ordering $\Re$ choosing a single option from two menus $A = \{a, b, c\}$ and $B = \{a, d\}$ of alternatives. In the case b and c are considered two undesirable options according to $\Re$, then the choice set from A should be considered better than that from B as long as it provides a minimum diversity among the reasonable[2] alternatives. In other words, we imagine that each individual in a given society selects what she most prefers to be or do (i.e. according to Sen (2006), she claims different meaningful lives), among all the possible opportunities a society offers. As a consequence, each individual is *identified* with the option she claims, and at the same time, her choice '*reveals*' her diversity from others. If someone chooses, for example, to eat certain food and to study for a doctorate in physics she will be identified as *diverse* with respect to another eating special dishes according to religious precepts and leaving school as soon as possible. As much as a society enhances *revealed diversity* among its members, it can be considered better in the sense that it allows more "significant choices with respect to various aspects of personal life" (Nozick, (1974)).

In other words, we value a society in which individuals make more heterogeneous claims as a better one. In our case, the importance of diversity comes directly from the value of freedom of choice, i.e. it depends on the liberty people have in their choice processes (see e.g. Sen (2006)). This paper is the first attempt to formalize the idea that a meaningful definition of diversity from a social point of view must be conditional to the individual freedom of choice. Evaluation of diversity depends on what individuals select as an opportunity when they claim their liberty, and not simply on the number of different options they have. We aim to study the diversity of a (free) society in which individual choose (their life plans) revealing indirectly their dissimilarities with the others.

The two primitives of our analysis are: an opportunity set (A) of non-rival and excludible opportunities, and a collection of n individual (linear) preference orderings[3] denoted by $(\Re)$. Each element in the latter selects a single most preferred option from the former. Hence, each pair $(A, \Re)$ originates a set of choices $C(A, \Re)$ of cardinality n. Nothing prevents the set $C(A, \Re)$ to include identical choices. We first analyze a criterion that compares each individual best choice with those of

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[1] A suggested sample of different categories of functionings *à la Sen*, each of which an individual can choose to practise in a (free) society, consists for example in claiming to be a European, an Italian citizen, a Tuscan with Spanish ancestry, a French resident, an economist, a man, a feminist, a strong believer in democracy, a defender of gay and lesbian rights and a nonbeliever in afterlife.

[2] The meaning of "reasonable" could be intended as in Jones and Sugden (1982).

[3] $(\Re)$ could be also be interpreted as the *multiple selves* of an individual rather then a set of *potential* preferences that she actually has in a society as in Jones and Sugden (1982)).

others within the same $C(A, \Re)$ in terms of their revealed diversity. In particular, we consider a single choice to be more dissimilar than another if the number of choices by others which are identical to the latter is lower than those identical to former. Thus, we introduce and characterize the so-called co-cardinality total ordering of dissimilarity, which can be seen as the dual (in the present more general setting) of the celebrated cardinality criterion characterized by Pattanaik and Xu (1990).

Then, we propose a diversity criterion that ranks different pairs of $(A, \Re)$. The ranking is induced by a family of frequency-based measures of revealed diversity. This class of evaluation functions is obtained by monotonic transformations of weighted averages of the evaluations of each single choice. Specifically, for any $(A, \Re)$, we average the dissimilarity of each individual (best) choice in A, normalized by the number of individuals in the reference population. The diversity ranking we obtain is therefore a complete preorder. The characterization of this criterion relies on a new property that rules how the evaluation of $(A, \Re)$ changes when a single element in the preference profile $\Re$ changes. In particular, if an individual, whose preferences in $\Re$ are changed, now selects a different option allows more revealed diversity (in the sense explained above) than the one selected before, then the aggregate diversity must increase.

A result of our analysis is that an element of the family of evaluation functions we study is the Shannon entropy measure (see also subsection 3.2 below). For its extreme computational convenience for applications, Shannon's entropy is indeed the most widely used index of diversity (see e.g. Hershey (2009) and Gravel(2008)). However, its axiomatic characterizations usually rely on 'the informativeness of a pair of independent distributions being the sum of their respective levels of informativeness' (see e.g. Theil (1967)), that is a requirement that lacks of compelling justifications when Shannon's entropy is used to measure diversity of actual choices. Indeed, the selection of an option from two independent pairs of $(A, \Re)$ typically does not coincide with the choice from the two sets merged together, unless a particular and arbitrary restriction on the preference domain is adopted. The present paper avoids this major drawback by proposing (indirectly) an alternative characterization of the Shannon entropy that is conducive to a fruitful approach to the issue of diversity in economic environments.

It is worth noticing here that the introduction of frequencies for individual choices is quite novel in economic literature on axiomatic measurement of diversity.[4] It allows us to consider the role of preferences in each single option's contribution to diversity enhancement and prevents Sen's (1991) critique of the so-called objective rankings of opportunity sets. In fact, since the work of Sen (1990), (1991), individual preferences are considered to have a vital role in judgements regarding freedom and/or opportunity. However, the origin of such preferences may seem quite arbitrary if not based on actual choices.[5] In the present work, we endogenously justify

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[4]The more traditional approach focuses on the objective diversity measurement of the options in a given menu. Indeed, according to Gravel (2008), we can distinguish at least three approaches sharing this last view: aggregate cardinal dissimilarity (see Weitzman (1992), (1998); Bossert et al. (2003), Van Hees (2004)), aggregate ordinal dissimilarity (Pattanaik and Xu (2000), Bervoets and Gravel, (2004)), and the valuation of realized attributes (Nehering and Puppe (2002)).

[5]Jones and Sugden (1982) and Sugden (1998) consider *potential preferences*, namely the set of "all possible preference orderings that an individual might reasonably have". For instance, Pattanaik and Xu (1998) proposed a model in which individuals are endowed with a given set

our set of complete and transitive orderings by assuming that they are *revealed* by a decisional process of choice, namely there exists a one-to-one correspondence between the rules of individual choices that satisfy certain plausible properties and the class of preferences we use to rank sets of opportunities in terms of their diversity. The choices made by people from some set X of all possible options (e.g. alternative life-plans) have a rational explanation, or rationalizing ordering (that is a linear order) such that for any $A \subseteq X$, an individual's choice from A is the best element in A according to that ordering. In other words, whatever the choice $a \in A$, it can be explained by an ordering, the maximization of which is consistent with the individuals' behavior (see Kalai, Rubinstein and Spiegler (2002) and in particular Aizerman and Malishevski (1981)). We therefore focus on the foregoing motivation to justify the use of preferences in our setting.

The remainder of our paper is organized as follows. In the next section, we introduce the notation, definitions, and axioms that provide a first result on how to rank options from a set of choices in terms of their (relative) dis-similarity. Section 3 contains our main result and some prominent examples. Section 4 concludes.

2. How to compare individual choices in terms of diversity

2.1. **Notation and definitions.** Let X be the universal set of options, assumed to be finite, and $N = \{1, ..., n\}$ be the set of individuals of a given population. We denote with $\wp(X) = 2^X \setminus \{\emptyset\}$ the set of all non-empty subsets of X. The elements A, B, C, etc. of $\wp(X)$ are the different feasible sets. A choice correspondence denoted $c : \wp(X) \rightrightarrows \wp(X)$ assigns a (sub-)set of choices to each set, formally for any $A \in \wp(X)$, $c(A) \subseteq A$. The choice is called *singular* (or equivalently *univalent* or *resolute*) if for any $A \in \wp(X)$ only one option is chosen from A, i.e. the cardinality of $c(A)$, denoted $|c(A)|$, is equal to 1. Let $c_i(\cdot)$ with $i \in \{1, ..., n\}$ be the *singular* choice function of an individual i selecting the maximal element from a set. A choice set $C(A)$ is the set of all maximal elements of a set A which are selected by the agents. In the present framework, we allow an element of any given set A to occur a finite number of times in $C(A)$. $C(A)$ may include as many copies of the same element as the number of individuals for which it is a maximal element.[6] Therefore, a generic choice set $C(A)$ may have the mathematical structure of a *multiset*,[7] because individuals may select the same option from A, and thus the same option could be allowed to appear more than once in $C(A)$. Thus, the number of elements in $C(A)$, i.e.$|C(A)|$, is equal to the number of individuals in the population considered. The present restriction for a choice set to be a multiset strongly suggests that in our model, options are best regarded as *non-rival*, which means that $a \in A$

of *reasonable preferences*. However, both approaches take the relevant preference orderings as exogenously determined. Dietrich and List (2012) claim that "an agent's preferences are based on certain 'motivationally salient' properties of the alternatives over which preferences are held". In few words, they explain a given set of preferences using other, let us say, deeper preferences, which, in our opinion, need a further recursive explanation.

[6]For instance, if we take a set $A = \{a, b, c, d, e, f\}$ and a population $N = \{1, 2, 3, 4\}$ such that individuals $1, 2, 3$ selects as their option a and individual number 4 chooses c, then, the resulting choice set will be $C(A) = [a, a, a, c]$, where the square brackets denote a set in which the same element can occur several times.

[7]Note that a finite multiset on X is defined as a function $m : X \to \mathbb{Z}_+$ such that $\sum_{x \in X} m(x) < \infty$, i.e. each member of a multiset has a multiplicity, which is a natural number indicating (loosely speaking) how many memberships it has in the multiset. Like sets, multisets support operations to insert and withdraw items and the basic set operations of union, intersection, and difference.

may be a bundle of rights and basic liberties (or functionings) that everybody could claim in order to live a life worth living. This interpretation is consistent with our idea that, in a democratic society, an individual simply chooses her life-plan as opposed (therefore diverse) to that of others without preventing others from doing so and she thus reveals her diversity from the rest of the population. We also denote with a_i the choice of individual $i \in N$ from the set A, namely $a_i = c_i(A) = \{a\}$. Therefore, $C(A) = [c_1(A), ..., c_i(A), ..., c_n(A)]$ or equivalently $C(A) = [a_1, a_2, ..., a_n]$. If there is only one individual selecting an option $a \in A$, we say tha a is an *isolated* choice (IC).

2.2. **Axioms and a preliminar result.** We now proceed with our analysis of revealed diversity by introducing a notion of *similarity* of items in a choice set. In other words, given any choice set $C(A)$, we define a binary relation $\succeq$, over the set (of all possible) personal choices, which establishes that for any $a_i, a_j \in C(A)$, a_i provides at least as much *dissimilarity* as a_j, i.e. $a_i \succeq a_j$.[8] In particular, we characterize the following measure of dissimilarity of choices $\succeq_d$:

Definition 1. *Let $A \in \wp(X)$ and $C(A)$ the corresponding choice set, for any a_i,$a_j \in C(A)$, we say that*

$$a_i \succeq_d a_j \quad \text{if and only if} \quad n_i(A) \geq n_j(A) \tag{2.1}$$

where for each $a_s \in C(A)$, $n_s(A) \equiv |\{a \in C(A) \mid a \neq a_s\}|$.

In words, we say that, given a choice set $C(A)$, the choice a_i provides at least as much dissimilarity as a_j if and only if the number of individuals (or chosen options) choosing an option different from a_i is not smaller than the number of individuals choosing an option different from a_j.[9] The ordering $\succeq_d$ is the so-called co-cardinality total preordering induced by our notion of dis-similarity and relies on the information provided by a cardinally meaningful numerical distance between the objects of the choice set in question.

We now characterize $\succeq_d$ using the following list of suitable properties:

Total Preorder - T. $\succeq$ is a total and transitive binary relation.

Indifference - I. Given $C(A)$ and for any $a_i, a_j \in C(A)$ that are IC, $a_i \sim a_j$.

Dominance - D. Given $C(A)$, for any $a_i \in C(A)$ that is IC and any $a_j \in C(A)$ that is not, $a_i \succ a_j$.

A finite l-partition $\mathcal{F}_M = \left(F_M^1, ..., F_M^l\right)$ of a multiset M is a collection of multi-sets, called *blocks*, whose multiset union (i.e. union with repetitions) is M (see e.g. [1]). Given a non-empty choice (multi-)set $C(A)$ whose cardinality is at least five and that contains at least two different options,

Independence - N. For any $a_i, a_j \in C(A)$ and any 2-partition $\mathcal{F}_{C(A)} = \left(F_{C(A)}^1, F_{C(A)}^2\right)$ of $C(A)$ such that:

$$a_i \succeq a_j \text{ for } a_i, a_j \in F_{C(A)}^1, \text{ and } a_i \sim a_j \text{ for } a_i, a_j \in F_{C(A)}^2,$$

[8]Note that $\succ$ and $\sim$ represent the asymmetric and symmetric parts of $\succeq$, respectively.

[9]In terms of our previous example with $C(A) = [a, a, a, c]$, c (that among other things is a IC), is a more dissimilar choice than a or differently individual number 4 has undertaken a choice different from the other members of the society.

we have that $a_i \succeq a_j$.[10]

A similarity ranking is induced by a reflexive (any set is at least as similar as itself), symmetric and transitive binary relation. Because, for example, a social decision-maker or government agency that intends to measure the degree of biodiversity of different ecological environments must be able to establish that one environment is more (or less) similar than another or that both have the same level of similarity, we assume that our ordering is also complete. Hence, the first axiom has its own rational.

The other three axioms have a natural interpretation. Indifference establishes that any singleton set provides the same dissimilarity. This property is satisfied by most indices used in current economic literature. However, some scholars claim that conceptions of dissimilarity that focus on the attributes of the objects in a set rather than on the objects themselves have no reason to observe this property: in principle, there is no reason to consider two ecological environments with only mosquitoes or human beings as indifferent in terms of dissimilarity they provide. This criticism does not apply to our framework. We compare options that are bundles of (*positive*-valuable) items, such as individual rights and personal liberties, on which there are no *a priori* preferences. In other words, in the present setting, saying that the choice of being, for example, a painter by a person who would like to be different from others is better than an analogous choice of another of being a lawyer is totally arbitrary or requires (a class of meta-) preferences that are, difficult to justify and on the whole unnecessary for the aim of the present analysis of diversity. We can therefore rely on this axiom.

Since isolated choices represent sets of maximal dissimilarity, Dominance requires that sharing a choice with others leads to a set that is worse in terms of dissimilarity according to $\succ$ than any choice taken in isolation. As dominance-type axioms tend to rule out rankings of sets that are based on 'total-goodness' criteria with respect to $\succ$ (see Fishburn (1988)), the dominance axiom appears to be a plausible requirement in the present work.

Independence makes it possible to consider unions of choice (sub-)sets so that implications can be derived for potential choices ruled by I or D under larger sets. More specifically, Independence concerns the order of two options obtained after merging two different choice sets. In such a case, indifference between options in one of the two choice sets is neutral for the determination of the order of the corresponding options in the final choice set.

The above axioms fully characterize the total preordering $\succeq_d$, as the following result shows:

Proposition 1. *Let $\succeq$ be a complete preorder on $C(A)$, Then $\succeq$ satisfies I, D, N if and only if $\succeq = \succeq_d$.*

Rule 2.1 differs from the cardinality total preorder rule characterized by Pattanaik and Xu (1990). Indeed, it compares items of the same choice multiset, rather than sets, and, more important, it is the first step to precisely formalize the idea that having a large number of (similar) alternatives available does not provide

[10]In order to fix the concept of multiset partition, consider for instance the following $C(A) = [a, a, a, b, b, c, e]$ and one of its possible partitions into 2-blocks with repeated blocks and elements allowed, namely: $[[a, a, b, c], [a, b, c, e]]$. In the first block, b is more dissimilar than a, while in the second one they are indifferent in terms of dissimilarity.

a higher degree of freedom as long as the distinct options are similar. Definition 1 considers individuals' freedom to choose as an effective means to analyze their dissimilarity and consequently the diversity of the society represented by the choice set under consideration as a desirable feature in itself.

3. On the comparison of choice sets in terms of diversity

3.1. **More axioms and the main result.** In what follows, we compare pairs of choice (multi-)sets, that eventually differ in size, in terms of *aggregate revealed diversity*. In order to do so, we introduce a binary relation $\succeq$ such that, $C(A) \succeq C(B)$ means that the choice multiset $C(A)$ provides at least as much *aggregate diversity* as the choice set $C(B)$. In particular, we study the following prominent notion of aggregate diversity:

Definition 2. *For any $A, B \in \wp(X)$, let $C_N(A)$ and $C_{N'}(B)$ be the choice multisets of two different populations $N = \{1, ..., n\}$ and $N^{'} = \{1, ..., n'\}$ of individuals, $\succeq_D$ is an* aggregate diversity *total preorder, defined by the following rule:*

$$C_N(A) \succeq_D C_{N'}(B) \quad \text{if and only if} \quad D(C_N(A)) \geq D(C_{N'}(B)) \tag{3.1}$$

where for any $Z \in \wp(X)$ and $M = \{1, ..., m\}$, $D(C_M(Z)) = \sum_{i=1}^{m} n_i(Z)$, with $n_i(Z)$ the measure of dissimilarity inducing $\succeq_d$.

In words, for a given choice set $C(Z)$, $D(C(Z))$ is the sum of the measure of diversity $\succeq_d$ of *any* choice in $C(Z)$. The criterion underlying 3.1 takes into account the degree of dissimilarity between options. It establishes that the diversity of a (choice) set is obtained by aggregating the dissimilarities between the elements of that set. We now examine under what circumstances this is true by axiomatically characterizing $\succeq_D$ as follows:

Replication Principle - (R). For any $A \in \wp(X)$, let $C(A)$ be the corresponding choice (multi-)set and let

$$[C(A)]^t \equiv \left[\underbrace{C(A)}_{1} \cup \underbrace{C(A)}_{2} \cup, ..., \underbrace{C(A)}_{t-1} \cup \underbrace{C(A)}_{t}\right]$$

denote the t-replication of $C(A)$, then $[C(A)]^t \sim C(A)$.

Option Anonimity - (OA). For any $A, B \in \wp(X)$, with $B = [A \backslash \{a\} \cup \{b\}]$, where $b \in X \backslash A$. Then, $C(A) \sim C(B)$ if $c_i(B) = \{b\}$ for any $i \in N$ such that $c_i(A) = \{a\}$.

Anonymity - (A). For any $A \in \wp(X)$, let $C(A)$ be the correspondent choice (multi-)set. Then, for any permutation $\pi : N \to N$ such that $\pi C(A) = \left[c_{\pi(1)}(A), ..., c_{\pi(n)}(A)\right]$, we have that

$$C(A) \sim \pi C(A).$$

Relevance of Dissimilar Choices - (RIC). For any $A \in \wp(X)$, let $C(A)$ be the correspondent choice (multi-)set and $C^{'}(A) = [C(A) \backslash c_j(A) \cup c_h(A)]$ with $c_h(A) \succ_d c_j(A)$, Then,

$$C^{'}(A) \succ C(A).$$

Independence of Balanced Choice Substitutions - (IBCS). For any $A, B \in \wp(X)$, $i \in N$, let $C(A)$ and $C(B)$ be the corresponding choice (multi-)sets and $C^{'}(A) = [C(A) \backslash c_i(A) \cup \widehat{c}_i(A)]$, $C^{'}(B) = [C(B) \backslash c_i(B) \cup \widehat{c}_i(B)]$ such that $c_i(A), c_i(B)$ are not IS and $\widehat{c}_i(A), \widehat{c}_i(B)$ are either IS or such that $\widehat{c}_i(A) \succ_d c_i(A)$ and $\widehat{c}_i(B) \succ_d c_i(B)$. Then,

$$C(A) \succeq C(B) \text{ if and only if } C^{'}(A) \succeq C^{'}(B).$$

The Replication Principle just states that the aggregate diversity has to be neutral with respect to the number of individuals, i.e. the diversity of a given choice set does not change if we consider a (t-fold) repetition of its elements. Option Anonymity implies that the substitution of a single option, which does not affect the distribution of the choices in a given choice set, does not modify the value of the aggregate diversity of the new choice set. Anonymity implies that the permutation of the individuals does not affect the distribution of the choices in a given choice set and therefore does not modify the value of the aggregate diversity. The RIC axiom rules instead the changes in aggregate diversity after a single individual choice substitution. The last axiom establishes the set of all ordering-preserving transformations on the choice profiles.

We are now ready to state that:

Proposition 2. *Let $\succeq$ be a complete preorder, then $\succeq$ satisfies R, OA, A, RIC and $IBCS$ if and only if $\succeq = \succeq_D$.*

This result on ranking sets of opportunities in terms of the diversity revealed by individual choice captures the freedom of a social structure at an abstract level: a society that allows more pluralistic choices can be considered better than another in terms of the freedom/diversity it provides to its members.

4. Appendix: Proofs

Proof of Proposition 1. ($\Leftarrow$) That $\succeq_d$ be a total preorder and satisfy I, D, N is trivial;

($\Rightarrow$) Suppose $\succeq$ be a complete and transitive binary relation satisfying I, D, N. Then, take an $A \in \wp(X)$ and a population of n individuals. For any $C(A)$ that, in order to avoid trivial qualification, has cardinality at least five and contains at least two different elements, take $a_i, a_j \in C(A)$ such that $a_i \succ a_j$ but $n_i(A) < n_j(A)$. The fact $n_i(A) < n_j(A)$ implies that the number of individuals choosing the same option of individual i is greater than the number of individuals choosing the option chosen by j. Suppose then without loss of generality that $n_i(A) = s < t = n_j(A)$ with $s \geq 2$. Construct a 2-partition $\mathcal{F}_{C(A)} = \left(F^1_{C(A)}, F^2_{C(A)}\right)$ of $C(A)$, such that for some $a_i, a_j \in F^1_{C(A)}$ are both IC. Hence, according to axiom I, $a_i \sim_1 a_j$, where $\sim_1$ denotes the restriction of $\sim$ to $F^1_{C(A)}$. Since $n_i(A) = s < t = n_j(A)$ and $s = 2$ we have that $a_i, a_j \in F^2_{C(A)}$ is such that a_j is IC and a_i is not IC. Thus, by D, $a_i \prec_2 a_j$. Hence, by N, we obtain that $a_i \prec a_j$ and we have a contradiction. In the case in which $s > 2$ construct a ℓ partition and iterate the application of axiom N.

Now, let $a_i \sim a_i$ but suppose $n_i(A) \neq n_j(A)$. That is, assume without loss of generality that in particular $n_i(A) < n_j(A)$. Repeating the same argument above we get a contradiction.

Conversely, suppose that $n_i(A) > n_j(A)$ but $a_i \prec a_j$. According to the axiom D, the fact that $a_i \prec a_j$ implies, that a_j could be a IC but a_i is not, hence by definition of $n_s(A)$ we have a contradiction. Again, suppose that $a_i \prec a_j$, then by axiom N there exists a l-partition such that for some $k < l$ $a_i \sim_k a_j$ and $a_i \prec^{l-k} a_j$ for $(l-k)$. Now, without loss of generality let $|c_i(A)|_k = |c_i(A)|_k \geq 1$ while $t = |c_j(A)|_{l-k} > |c_i(A)|_{l-k} = s$. Hence, a contradiction.

Suppose now $n_i(A) = n_j(A)$ but $a_i \nsim a_j$ and in particular that $a_i \prec a_j$. A repetion of the same argument above entails a contradiction and completes the proof.[11] □

Proof of Proposition 2. To check that $\succeq_D$ is a total preorder and satisfies R, OA, A, RIC and $IBCS$ is straightforward.

Conversely suppose that the total preorder $\succeq$ satisfies R, OA, A, RIC and $IBCS$, then take $A, B \in \wp(X)$, two different size populations $|N| = n$, $|N'| = m$ and define with $C_N(A)$ and $C_{N'}(B)$ the corresponding choice (multi-)sets. By R, we have that $C_N(A) \sim [C_N(A)]^m$ and $C_{N'}(B) \sim [C_{N'}(B)]^n$. Therefore, if $C_N(A) \succeq C_{N'}(B)$ by transitivity we obtain that $[C_N(A)]^m \succeq [C_{N'}(B)]^n$. Observe now that for any two choice (multi-)sets $C(E) = [e, ..., e]$ and $C(F) = [f, ..., f]$, i.e. such that $c_i(E) = \{e\}$ and $c_i(F) = \{f\}$ for any $i \in N$, $C(E) \sim C(F)$, $D[C(\cdot)]$ takes its minimum value and $D[C(E)] = D[C(F)] = 0$ by construction. Thus, for any positive integer l, by a suitable l-repeated applications of IBCS to $C(E) \sim C(F)$, we obtain two new choice (multi-)sets $C'(E)$ and $C'(F)$ such that $C'(E) \sim C'(F)$ and, again, $D[C'(E)] = D[C'(F)]$. On the other hand, for any integer k and any $E \in \wp(X)$ such that $D[C(E)] = k$, consider $C(E) = [e, ..., e]$, i.e. such that $c_i(E) = \{e\}$ for any $i \in N$. Then, a finite set of substitutions $c_i(E) \succ_d e_i$, for some $i \in N$, produces the requested result. Thus, we conclude that any two choice (multi-)sets with the same aggregate diversity are indifferent.

So, consider again $[C_N(A)]^m \succeq [C_{N'}(B)]^n$ but suppose $D([C_N(A)]^m) < D([C_{N'}(B)]^n)$. By a repeated application of RIC to $[C_N(A)]^m$, we obtain a $[C'_N(A)]^m \succ [C_N(A)]^m$ such that $D\left([C'_N(A)]^m\right) = D([C_{N'}(B)]^n)$ that means that $[C'_N(A)]^m \sim [C_{N'}(B)]^n$, hence a contradiction.

On the other hand, suppose $[C_N(A)]^m \succeq_D [C_{N'}(B)]^n$, i.e. $D([C_N(A)]^m) \geq D([C_{N'}(B)]^n)$, and consider $C(W) = [c_1(W), ..., c_n(W)]$ such that $c_i(W) \neq c_j(W)$ for any $i, j \in N$. Observe that $C(W) \sim C(W)$ by reflexivity, then by AN, $IBCS$ and OA applied on $C(W) \sim C(W)$, we obtain $[C_N(A)]^m \sim [C_{N'}(B)]^n$. A further application of RIC on $[C_N(A)]^m$ entails the desired result. □

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[1] Bender, E. A. (1974) - Partitions of multisets - *Discrete Mathematics* 9, pp 301–311.Diversity, preferences and entropy

As a final remark we point out the connection between the class of evaluation functions that induces our diversity criterion (**??**) and the classical entropy measure advocated by Suppes (1996) and Erlander (2005) as a suitable tool for ranking

[11] Since the characterization of $\succeq_d$-trivially speaking- mirrors the characterization of the cardinality total preordering of opportunity sets due to Pattanaik and Xu (1990), we do not provide examples of the independence of the axioms used here, but can supply them on request.

opportunity sets in terms of freedom of choice. In fact, for any choice set $C(Z,\Re)$, the Shannon entropy measure, denoted as $Ent(\cdot)$, belongs to the class of frequency-based functions in (**??**). To show that it is enough to write $Ent(\cdot)$ as a frequency-weighted average of the order-preserving images of $d([z_i], C(Z,\Re))$ according to $-\log(1-x)$, namely:

$$Ent(C(Z,\Re)) = -\frac{1}{|C(Z,\Re)|}\sum_{i=1}^{n}\log\left(1-\frac{d([z_i],C(Z,\Re))}{|C(Z,\Re)|}\right). \tag{4.1}$$

In particular notice that order-preserving transformation of $d([z_i], C(Z,\Re))$ equally satisfy the relation $\succeq_d$defined over $(Z,\Re)$. Therefore Theorem 2 equally applies to 4.1. Shannon entropy has been widely used in biology to measure the diversity of ecosystems, since entropy is a measure of the "disorder" of a system. Translated into our setting, a set of opportunities that is maximally "disordered", namely has the greatest variety of dissimilar options, is considered maximally diverse. Note that Suppes (1996) and Erlander (2005) proposed a entropy-based measure of freedom of choice, but did not characterize it axiomatically. Our work could also be seen as the first axiomatic foundation for using entropy as a measure of diversity of choices. Suppes (1996) and Erlander (2005) motivated application of this measure by stochastic utility theory of logit models (see e.g. MacFadden (1974)).[12] However, the usual entropy interpretation and its well-known characterizations in physics and biology cannot directly be applied in an economic environment. Indeed, additivity[13], the key-property of almost all entropy characterizations, does not find a proper meaning in the economic context of revealed diversity unless we severely restrict the domain of individual preference orderings. Two distinct populations, choosing their best options from two different opportunity sets such that the resulting choice sets have a null intersection, will not typically select the same options when both populations and opportunity sets are merged together. In other words, it is not generally true that the entropy of choices satisfies additivity once the whole set of individuals has to select from the union of the two opportunity sets. This only happens in some very special cases after appropriate restriction of individual preference orderings. Our axiomatic method avoids this difficulty, making entropy a measure applicable to the context. In fact, joint application of the *preference substitution* axiom with *option anonymity* and the *replication principle* shows the direction in which the aggregate diversity evaluation of a generic $C(Z,\Re)$ changes after a single change in the preference profile (see Example 1).

5. Concluding note

In the present paper, we have explored the problem of ranking opportunity sets (the elements of which could be interpreted as bundle of rights and basic liberties), in terms of their diversity after the individuals (with well-defined preference profiles) of a population have selected their best choice. Since the choice concerns various aspects of personal life, it *reveals* the diversity of people in a society. A

[12]Indeed, in that perspective, the utility function is the propensity to choose and no longer a deterministic device as in standard utility theory. The analysis relies on the concept of statistical equilibrium as defined in e.g. Foley (1994).

[13]The entropy of a joint distribution of two variables is bounded (or equal to in the case of independent variables) from above by the sum of the entropies of the two distributions.

society that enhances (more) *revealed diversity* among its members can be considered better than a society where individuals make homogeneous claims, because diversity draws its value from greater freedom of choice (see e.g. Sen (2006)). If the set of opportunities a society provides to its members contains only one suitable option, 'human identities are formed by membership of a single social group' (see Sen, (2006)) and 'everyone is locked up in tight little boxes from which she emerges only to attack one another' (see Sen (2006)). "The prospects of peace, tolerance, freedom and democracy in the contemporary world may well lie in the recognition of the plurality (hence *diversity*) of our identities, where personal identity must be understood as an extension of one's own choice of being someone or doing something" (Sen (2006)). This study was devoted to providing a rationale for this insight, in an attempt to open new research perspectives in the analysis of freedom of choice and (individual) diversity.

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($\Rightarrow$) To check that $\succeq_D$ is a total preorder and satisfies R, A and $P.3$ is straightforward. To show that $\succeq_D$ also satisfies $P.1$ and $P.2$, take any $C(A,\Re) \in \aleph$ and for any $[a_i]$,$[a_j] \in C(A,\Re)$ suppose that $k_i = |[a_i]|$, $k_j = |[a_j]|$ and $|\Re| = m$ without loss of generality. By definition:

$$Ent(C(A,\Re)) = \log m - (1/m)\left[k_i \log k_i + k_j \log k_j + \mathbf{k}\right],$$

where $\mathbf{k}$ is a real number depending on both the number and distribution of the choices outside the set $\{[a_i] \cup [a_j]\}$. Now take $\Re' \in \wp(\mathbf{R})$ such that $\Re' = \Re \backslash \Re_j \cup \Re_h$ with $\Re_h \in \mathbf{R}$ and compute $Ent(C\left(A,\Re'\right))$. In the case $a_h = a_i$, we have:

$$E(C\left(A,\Re'\right)) = \log m - (1/m)\left[(k_i+1)\log(k_i+1) + (k_j-1)\log(k_j-1) + \mathbf{k}\right],$$

in which both the number and distribution of the choices outside the set $\{[a_i] \cup [a_j]\}$ are unaffected by the preference substitution. Thus, the difference $E(C\left(A,\Re'\right)) - E(C(A,\Re))$ is positive whenever

$$(1+k_i)\ln(k_i+1) - k_i \ln k_i < (k_j-1)\ln(k_j-1) + k_j \ln k_j \tag{5.1}$$

which is always true when $k_i < k_j - 1$ given that $(1+x)\ln(x+1) - x\ln x$ is monotonically increasing in x. If $[a_i] \succ_d [a_j]$ then $k_j > k_i$ so that $k_j - 1 \geq k_i$ (remeber $k_i, k_j \in N$). Hence, $C\left(A,\Re'\right) \succeq C(A,\Re)$, as required by $P.1$. If $[a_i] \sim_d [a_j]$, then $k_j = k_i$ and $E(C\left(A,\Re'\right)) < E(C(A,\Re))$ and therefore $C\left(A,\Re'\right) \prec C(A,\Re)$ as required by $P.2$. We therefore conclude that $\succeq_D$ also satisfies weak dominance and strict dominance in P.

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